Solution-Mid-Exam

(1). (a) True
Justification: If $p : E \to X$ is a covering, then $p_* : \pi_1(E, e) \to \pi_1(X, p(e))$ is injective. By hypothesis, $p_* : \pi_1(E, e) \to \pi_1(X, p(e))$ is an isomorphism. This implies that p is a 1-sheeted covering and hence homeomorphism.

(b) False
Justification: Let X be the quotient space of $\mathbb{S}^2$ obtained by identifying $(0, 0, 1)$ with $(0, 1, 0)$. There exists a cell complex structure on $\mathbb{S}^2$ consisting of two 0-cells ($(0, 0, 1)$ and $(0, 1, 0)$), one 1-cell, and one 2-cell (the remainder of the surface). By identifying $(0, 0, 1)$ and $(0, 1, 0)$ to a single point p, we get a corresponding cell complex structure of X consisting of:

- 0-skeleton: one point p;
- 1-skeleton: two 1-cells, α and β, with their endpoints identified to p;
- 2-skeleton: two 2-cells, with their boundaries identified to the path $\alpha \cdot \beta$.

Hence the 1-skeleton Y is homeomorphic to the wedge sum of two copies of $\mathbb{S}^1$, and $\pi_1(Y, p)$ is free with generators $a = [\alpha]$ and $b = [\beta]$. The space X is obtained from Y by adding 2-cells, hence $\pi_1(X)$ is the quotient of $\pi_1(Y)$ by the relations $\partial e^2 = 1$ for each 2-cell e^2. In our case, the two 2-cell give the same relation $ab = 1$ and we have:

$$\pi_1(X) =< a, b \ \mid \ ab = 1 >=< a >= \mathbb{Z}.$$

(c) True
Justification: Let $\phi : F_2 = \mathbb{Z} * \mathbb{Z} =< a, b >\to S_3$ be the homomorphism satisfying $\phi(a) = (12)$ and $\phi(b) = (123)$. By the classification of covering spaces, there exists a covering $p : E \to \mathbb{S}^1 \vee \mathbb{S}^1$ which corresponds to the subgroup $Ker(\phi) \subseteq \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong F_2$. Note that

$$Cov(E/\mathbb{S}^1 \vee \mathbb{S}^1) \cong F_2/Ker(\phi) \cong S_3.$$

(d) True
Justification: If $p : E \to B$ is the universal cover, then for every point

$x \in B$, we have an evenly covered neighborhood U_x of x. The inclusion $i : U_x \hookrightarrow B$, by definition, lifts to E, so

$$i_*(\pi_1(U_x, x)) \subseteq p_*(\pi_1(E, \tilde{x})) = (1).$$

Therefore i_* is the trivial map. This implies that B is semi-locally simply connected.

(2). The function $H_1 : Y \times I \to Y$ given by

$$H_1 : ((x, y), t) = (x, (1 - 2t)y)$$

for $t \in [0, 1/2]$ and

$$H_1((x, y), t) = (2(1 - t)x, 0)$$

for $t \in [1/2, 1]$ is a homotopy between the identity map id_Y and the constant map $c_{(0,0)}$ at $(0, 0)$, since $H_1 : Y \times I \to Y$ is continuous by Gluing Lemma, and

$$H_2((x, y), t) = (1 - t)(0, 0) + t(0, 1)$$

is a linear homotopy between the constant map $c_{(0,0)}$ at $(0, 0)$ and the constant map $c_{(0,1)}$ at $(0, 1)$. Then we define

$$F((x, y), s) = H_1((x, y), 2s)$$

for $s \in [0, 1/2]$ and

$$F((x, y), s) = H_2((x, y), 2s - 1)$$

for $s \in [1/2, 1]$ and represents a homotopy between the identity map id_Y and the constant map $c_{(0,1)}$ at $(0, 1)$.
To establish the second claim, we assume, on the contrary, that there is a homotopy $F : Y \times I \to Y$ such that $F(q, 0) = q$, $F(q, 1) = p = (0, 1)$ and $F(p, t) = p$, for all $t \in I = [0, 1]$ and all $q \in Y$. Note that $Y \times [0, 1]$ is compact and F is uniformly continuous. Consequently a $\delta > 0$ exists such that $|F(q, t) - p| < \frac{1}{2}$ for all $q \in Y$ and $|p - q| < \delta$ and all $t \in [0, 1]$. Fix $q_0 = (\frac{1}{n_0}, 1)$ such that $n_0 > \frac{1}{\delta}$. Since the set $\{F(q_0, t) \mid 0 \leq t \leq 1\}$ is connected and moreover $F(q_0, 0) = q_0$ and $F(q_0, 1) = p$, there exists a $t_0 \in [0, 1]$ such that $F(q_0, t_0) = (\frac{1}{n_0}, 0)$. Thus $|F(q_0, t_0) - p| > 1$, contradiction.

(3). (a)$\Rightarrow$ (b) Let $H : \mathbb{S}^n \times I \to X$ be a homotopy of $f : \mathbb{S}^n \to X$ to a constant map. Then H defines a map $\overline{H} : C\mathbb{S}^n \to X$ such that $\overline{H} \circ q = H$, where CX is the quotient space of $X \times I$ obtained by identifying the subspace $X \times 0$ to a single point and $q : \mathbb{S}^n \times I \to C\mathbb{S}^n$ is the quotient map. Now consider the homeomorphism $\overline{P}^{-1} : \mathbb{D}^{n+1} \to C\mathbb{S}^n$ induced by the map $P : \mathbb{S}^n \times I \to \mathbb{D}^{n+1}$ given by $(x,t) \to tx$. The composite $\bar{f} = \overline{H} \circ \overline{P}^{-1}$ will do the job.
(b)$\Rightarrow$ (c) Suppose f can be extended to a map $\bar{f} : \mathbb{D}^{n+1} \to X$. Let $F : \mathbb{D}^{n+1} \times I \to \mathbb{D}^{n+1}$ be the relative homotopy of the identity map with the constant map x_0 viz., $(x,t) \to (1-t)x + tx_0$. Take the composite $\bar{f} \circ F : \mathbb{S}^n \times I \to X$. (c)$\Rightarrow$ (a) Obvious.

(4). The covering projection $p : E \to X$ is called a regular covering (or a Galois covering or a normal covering) if the subgroup $p_*(\pi_1(E, \tilde{x}))$ is normal in $\pi_1(X, x)$.

Assume first that p is regular, and let α be a closed path in X based at x having a closed lifting $\widetilde{\alpha}$ in E based at $\tilde{x}$. If $\widetilde{\beta}$ is a lifting of α with $\widetilde{\beta}(0) = \tilde{y}$, then $p(\tilde{x}) = x = p(\tilde{y})$ and the subgroups $p_*(\pi_1(E, \tilde{x}))$ and $p_*(\pi_1(E, \tilde{y}))$ are conjugate in $\pi_1(X, x))$. Since $p_*(\pi_1(E, \tilde{x}))$ is normal in $\pi_1(X, x))$, we have

$$p_*(\pi_1(E, \tilde{x})) = p_*(\pi_1(E, \tilde{y})).$$

Consequently,

$$[\alpha] = p_*(\widetilde{\alpha}) \in p_*(\pi_1(E, \tilde{y}))$$

and so there exists a loop $\widetilde{\gamma}$ in E based at $\tilde{y}$ such that $p_*([\widetilde{\gamma}]) = [\alpha]$. It follows that $p \circ \widetilde{\beta} = \alpha$ is homotopic to $p \circ \widetilde{\gamma}$ rel $\{0, 1\}$. By the Monodromy Theorem, we see that $\widetilde{\beta}(1) = \widetilde{\gamma}(1) = \tilde{y}$, and thus $\widetilde{\beta}$ is a closed path.
Conversely, suppose that either every lifting relative to p of a closed path in X is closed or none is closed. To see that p is regular, consider a subgroup $p_*(\pi_1(E, \tilde{x})) \subseteq \pi_1(X, x)$. Then a conjugate of $p_*(\pi_1(E, \tilde{x}))$ in $\pi_1(X, x)$ is the subgroup $p_*(\pi_1(E, \tilde{y}))$ for some $\tilde{y} \in p^{-1}(x)$. So it suffices to prove that

$$p_*(\pi_1(E, \tilde{x})) = p_*(\pi_1(E, \tilde{y}))$$

whenever $p(\tilde{x}) = p(\tilde{y})$. Let $[\widetilde{\alpha}] \in \pi_1(E, \tilde{x})$. By the Path Lifting Property of p, there is a path $\widetilde{\beta}$ in E with $p \circ \widetilde{\beta} = p \circ \widetilde{\alpha}$ and $\widetilde{\beta}(0) = \tilde{y}$. Since $\widetilde{\alpha}$ is a closed lifting of $p \circ \widetilde{\alpha}$, $\widetilde{\beta}$ must be closed (by our assumption). Thus $[\widetilde{\beta}] \in \pi_1(E, \tilde{y})$ and $p_*([\widetilde{\beta}]) = p_*([\widetilde{\alpha}])$. This implies that

$$p_*(\pi_1(E, \tilde{x})) \subseteq p_*(\pi_1(E, \tilde{y})).$$

Similarly

$$p_*(\pi_1(E, \tilde{y})) \subseteq p_*(\pi_1(E, \tilde{x})),$$

and the equality holds.

(5) Let $F_2 = \mathbb{Z} * \mathbb{Z}$ be the free product of two copies of $\mathbb{Z}$, the first copy generated by a and the second generated by b. Let $H = \ll \{a^4, ab, ba\} \gg$ be the normal closure of $\{a^4, ab, ba\}$ in F_2.
Claim:

$$F_2/H = K = \{H, aH, a^2H, a^3H\}.$$

We need only show that $(aH)K = K$ and $(bH)K = K$. The first case is clear. One need to focus on the second case. For example,

$$(bH)(H) = bH = ba^4H = baa^3H = baHa^3 = Ha^3 = a^3H \in K.$$

Other cases are similar. So, F_2/H has 4 elements. This implies that H has index 4 in F_2.